MATH1200E: A Note on Mathematical Induction

Tutor: Nathan Gold

October 28, 2017

One very important proof technique that comes up all the time when trying to prove some property about elements of a well-ordered set, that is, a set with the relation $\leqslant$ between elements, is the technique of *mathematical induction*. Where we will be seeing it most commonly used is to prove some statements about natural numbers $\mathbb{N}$, although as you continue studying mathematics, it will come up again and again when trying to prove some facts about iterations.

Induction is a direct proof method, not a proof by contradiction. The first time one sees an induction argument, it can be a little confusing, and you will often think, "What does that even show?". Once you get past this initial hurdle, you'll realise that induction is a perfectly well-defined proof technique, and one that can make life a lot easier when trying to prove that statements hold for natural numbers.

An induction argument is generally used to prove that a statement involving some $k \in \mathbb{N}$ holds for all $k \in \mathbb{N}$.A proof by induction argument proceeds in two steps:

1. **Base case**: Prove that the statement holds for the first natural number you are considering. Typically, $k = 0$, or $k = 1$.
2. **Induction**: Suppose now that the statement holds for some natural number $k = n$, and show that the statement holds for the $k = n + 1$ natural number. To do this, you often need to use what you proved back in the base case to extend things to the $k = n + 1th$ term.

The idea here is that you are proving that you can go as far as you wish, and the statement still holds. By way of an analogy, consider climbing up a ladder. The inductive argument shows that if you can walk onto the first step of the ladder (the *base case*), then you can walk up to each next step from the one you are currently on (the *induction*.)

Let's try to prove the following statement:

Proposition: Show that the sum of n natural number is,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}. \tag{1}$$

Proof. Now since we are being asked to prove this in a note about mathematical induction, you will naturally assume that you need to prove this inductively, however, when you encounter statements such as this in the wild, you will have no such hints. How to recognise when to prove something inductively is a skill you can develop over time by observing what the statement is actually asking you to show. This statement has all of the hallmarks of an inductive proof. You are being asked to show a general property about natural numbers $n \in \mathbb{N}$.

If we pause and think for a moment before launching into the induction, a naïve attempt is to launch right into the process and see what we can get. Let's do that for a few cases:

- $n = 0 : \frac{(0)(0+1)}{2} = 0.$
- $n = 1 : 0 + 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$
- $n = 2 : 0 + 1 + 2 = \frac{2(2+1)}{2} = \frac{2(3)}{2} = \frac{6}{2} = 3.$

So far it seems like things are working out, and the above formula (1) is believable. Now let's try to generalise this by induction.

Base case: The base case is when $k = 0$ or in the formula, $n = 0$: $\sum_{i=0}^{0} i = 0$. Since we already showed that this formula holds above, the base case has been established.

Induction: Now that the base case has been established, we proceed to make the *inductive hypothesis*, that is, that the statement holds for $k = n$. Thus, we claim that the formula is indeed true,

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{2}$$

Now, we need to show that the statement holds for $k = n+1$, and we are done. To do this, we will need to make use of the inductive hypothesis. Take $k = n+1$. Then,

$$\sum_{i=0}^{n+1} i = \underbrace{0 + 1 + 2 + \cdots + n}_{\sum_{i=0}^{n} i} + (n+1).$$

By the inductive hypothesis, the sum of the first $n+1$ terms (count them up and you will see we have $n+2$ terms here because we include 0) is $\frac{n(n+1)}{2}$, so

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= \underbrace{0 + 1 + 2 + \cdots + n}_{\sum_{i=0}^{n} i} + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{n^2 + 3n + 2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

after factoring the numerator. This is exactly what we wanted to show!

Hence, we have shown that the statement holds for the first (*base*) case, and we have shown that it holds for each subsequent step (from n to $n+1$). Thus, no matter what number you try, the statement will always hold. Therefore, our proof is complete. $\square$

A famous story in the histroy of mathematics is about the 19th century mathematician Carl Friedrich Gauss. Gauss was likely one of the smartest people to ever live, and there is no area of mathematics and the natural sciences that was not dramatically advanced by his work. The story goes that when Gauss was eight years old, his school teacher did not feel like teaching one day in class and assigned the students the problem of summing up the numbers from 1 to 100, that is, $1 + 2 + \cdots + 99 + 100$. Being a child prodigy, Gauss nearly immediately answered 5050, which is the correct answer. Let's use the above formula to prove him right!

$$\sum_{i=0}^{100} i = \sum_{i=1}^{100} i = \frac{100(101)}{2} = \frac{11000}{2} = 5050.$$

Gauss had realised this formula himself at eight.