Decimal expression of rational numbers with repeating zeros

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Here is proof of a homework problem in Liebeck (problem 6, chapter 3, 3rd edition) that may prove useful.

Problem: Let $\frac{m}{n}$ be a rational number in lowest terms. Prove that is has a decimal expression ending in repeating zeros $(0000\ldots)$ iff $n = 2^a 5^b$, for $a, b \in \mathbb{Z}$, where $a, b \geqslant 0$.

Proof. We have to prove an if-and-only-if statement, so we need to go in both directions.

($\Rightarrow$): Let $\frac{m}{n}$ have a decimal expression ending in repeating zeros, that is $\frac{m}{n} = a_0.a_1 \ldots a_k 00000$. Without loss of generality, take $a_0 = 0$.

We now need ot show that $n = 2^a 5^b$ for $a, b \in \mathbb{Z}$, with $a, b \geqslant 0$. We can drop the repeating zeros, so $\frac{m}{n} = 0.a_1 a_2 a_3 \ldots a_k$. Then,

$$\frac{a_1 a_2 a_3 \ldots a_k}{10^k} = \frac{a_1 a_2 a_3 \ldots a_k}{2^k 5^k}$$

where we cancel out factors of either 2 or 5 in each $a_1, \ldots a_k$.

($\Leftarrow$): Conversely, suppose that $\frac{m}{n}$ is such that $n = 2^a 5^b$ for $a, b \in \mathbb{Z}$, where $a, b \geqslant 0$. We now need to show that $\frac{m}{n}$ has a repeating 0's decimel expression.

$$\frac{m}{n} = \frac{m}{2^a 5^b} = \frac{m 5^{a-b}}{10^a}$$

which ends in repeating 0's.

Hence, $\frac{m}{n}$ has a decimal expression ending in repeating 0's iff $n = 2^a 5^b$, where $a, b \in \mathbb{Z}$, $a, b \geqslant 0$. $\square$