Math1200E Proposition 3.3 Proof - Repeating Decimals

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Here is a more detailed proof of Proposition 3.3 in your text.

Proposition 3.3: Suppose that $a_0.a_1a_2a_3\dots$ and $b_0.b_1b_2b_3\dots$ are two different decimal expressions of the same real number. Then one of these expressions ends in $9999\dots$ and the other ends in $0000\dots$.

Proof. The purpose of this proof is to show that really the only way that two decimal expressions of a number can ever be equal is if to have one of the expressions ending in repeating 9's, and the other ending in repeating 0's. What we are going to do is try to take some decimal expressions of these forms and show that this is indeed what we get.

Let $x \in \mathbb{R}$, and take two decimal expressions $x = a_0.a_1a_2a_3\dots$ and $x = b_0.b_1b_2b_3\dots$. Without loss of generality, let's first suppose that $a_0 = b_0 = 0$. This is only a matter of convenience, since we know the decimal expression of any real number may be written,

$$x = \sum_{i=0}^{\infty} \frac{a_i}{10^i} = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots$$

so we can always just add in any a_0 we want to get any other number we care about. As we have claimed that x has the decimal expressions, then $x = a_0.a_1a_2a_3\dots x = b_0.b_1b_2b_3\dots$.

Since we are assuming that the decimal exprssions are different, we know that at some decimal index the terms of the decimal expression are going to be different. Let's take that index to be k; that is, the decimal expressions agree for the terms up to and including $i = k-1$, and differ at $i = k$. Mathematically, this is,

$$0.a_1a_2a_3\dots a_{k-1}a_k\dots = 0.a_1a_2a_3\dots a_{k-1}b_k\dots$$

Notice that we have replaced the $b_1,\dots,b_{k-1}$ with the $a_1,\dots,a_{k-1}$, since we have assumed they agree.

Now, as is done in the text, we take that $a_k > b_k$ - this is our own choice, we could also take $a_k < b_k$. Since the terms of the decimal expressions can only be integers from 0 to 9, we have immediately that $a_k \geq b_k + 1$ - the a_k must be at least one bigger than the b_k term!

Now, we will *truncate* or *cut-off* the decimal expression of x in the a's and the b's. That is, we will go up to our point of interest, index $i = k$, and then change the terms of the decimal expression appearing after that to get an inequality we can deal with. So,

$$x \geq 0.a_1a_2a_3\dots a_{k-1}a_k0000\dots \tag{1}$$

x is greater here than this decimal expression because we are making all the a_i, $i > k$ 0, which will be less than the actual terms appearing in the decimal expression of x. We choose the 0's here because they are the smallest numbers we can take (this is why that is one of the decimal expressions we recover in the statement of the theorem). Conversely, we will make the b's expression bigger (or equal to),

$$x \leq 0.a_1a_2a_3\dots a_{k-1}b_k9999\dots.$$

Just like we did with the a_k, we set all the terms appearing after b_k to be 9's because that is the biggest number we can take for each position, necessarily making it greater than or equal to the correct decimal expression for x. Together we have,

$$0.a_1a_2a_3\dots a_{k-1}a_k0000\dots \leq x \leq 0.a_1a_2a_3\dots a_{k-1}b_k9999\dots$$

Now here is the part that may be confusing. We are going to use the fact that $0.9999\ldots = 1$, just shifted down further in the decimal expression; that is $0.00009999\ldots = 0.00001000\ldots$. In the context of this proof, this translates to

$$x \leq 0.a_1a_2a_{3k-1}b_k9999\ldots = 0.a_1a_2a_3\ldots a_{k-1}(b_k+1)000\ldots \tag{2}$$

Therefore, by looking at the decimal expression in equation (1) and comparing it to the second part of the expression in equation (2), we have $a_k = b_k + 1$. Therefore,

$$x = 0.a_1a_2a_3\ldots a_k000\ldots = 0.a_1a_2a_3\ldots a_{k-1}(a_k-1)999\ldots.$$

Thus, one decimal expression ends in repeating 9's, and the other ends in repeating 0's. □